SOLUTION OF REGULAR BEAM EQUATIONS IN ARBITRARY EMISSION CONDITIONS ON A CURVILINEAR SURFACE

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The regular beam equations are solved analytically for the case of emission from an arbitrary surface in conditions of total space charge (ρ-mode) and in a given external magnetic field $H \neq$ (§2); for temperature-limited emission (T-mode), in an external magnetic field H (§3); and for emission with nonzero initial velocity (§4). The emitter is taken as the coordinate surface $x^1 = 0$ in an orthogonal system x^i ($i = 1, 2, 3$), while the current density J and field ε on it are given functions $J(x^2, x^3)$, $\varepsilon(x^2, x^3)$. The solution is written as series in $(x^1)^\alpha$ with coefficients dependent on x^2, x^3, determined from recurrence relations. For emission in the ρ-mode and $H \neq 0$, $\alpha = 1/3$; for temperature-limited emission, $\alpha = 1/2$; with nonzero initial velocity, $\alpha = 1$. The results are extended to the case of a beam in the presence of a moving background of uniform density (5).

§1. **Fundamental equations.** A regular single-energy nonrelativistic beam of charged particles having fixed specific charge η of fixed sign, is described in the stationary case by a system of differential equations which can be written in the tensor form

$$g^{ik}v_iv_k + (u)^2 = 2\varphi, \qquad H^l = \frac{1}{\sqrt{g}} e^{ikl} \frac{\partial v_i}{\partial x^k},$$
$$\frac{\partial}{\partial x^i}(\sqrt{g} g^{ik} \rho v_k) = 0, \qquad \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g} g^{ik} \frac{\partial \varphi}{\partial x^k}\right) = \rho, \qquad (1.1)$$

where x^i ($i = 1, 2, 3$) is a curvilinear coordinate system, v_i are the covariant velocity components, φ scalar potential, ρ is the space charge density, and H^l are the contravariant components of the external magnetic field vector. The equations are written in the dimensionless variables $r°$, $V°$, $\varphi°$, $\rho°$, $H°$ (r, V, H are the moduli of the radius vector, velocity vector and external magnetic field vector)

$$r = ar°, \quad V = UV°,$$
$$\varphi = -\frac{U^2}{\eta}\varphi°,$$
$$\rho = \frac{U^2}{4\pi\eta a^2}\rho°, \quad H = \frac{cU}{\eta a}H°, \qquad (1.2)$$

after omitting the superscript indicating the dimensionless variable; a, U are constants with the dimensions of length and velocity respectively, and c is the velocity of light. In the first of Eqs. (1.1), u is the constant initial velocity of the particles on the surface $\varphi = 0$.

It will be assumed that $x^1 = 0$ is the equation of the emitter in the orthogonal coordinate system x^i ($i = 1, 2, 3$).

The magnetic field is assumed given. In the problems considered below, a knowledge of two components H^2, H^3 is sufficient, since H^1 can be found from Maxwell's equations. Only one further equation,

$$e^{ikl} \frac{\partial H_i}{\partial x^k} = 0, \qquad (1.3)$$

need be added to (1.1) in order to complete the system, since the fact that H is solenoidal follows from the conditions for the flow to be regular (the second of Eqs. (1.1), which says that the generalized momentum $P_i = v_i + A_i$, A is the vector potential).

Explicity, the conditions in question are

$$\frac{\partial v_2}{\partial x^3} - \frac{\partial v_3}{\partial x^2} = \sqrt{g} H^1,$$
$$\frac{\partial v_3}{\partial x^1} - \frac{\partial v_1}{\partial x^3} = \sqrt{g} H^2, \qquad \frac{\partial v_1}{\partial x^2} - \frac{\partial v_2}{\partial x^1} = \sqrt{g} H^3.$$

From the last two equations,

$$v_2 = \int \left(\frac{\partial v_1}{\partial x^2} - \sqrt{g} H^3\right) dx^1, \qquad v_3 = \int \left(\frac{\partial v_1}{\partial x^3} + \sqrt{g} H^2\right) dx^1;$$

substituting these values in the first,

$$\int \frac{\partial}{\partial x^i}(\sqrt{g} H^i)\, dx^1 = 0,$$

whence

$$\frac{\partial}{\partial x^i}(\sqrt{g} H^i) = 0,$$

as required. Thus the flow cannot be regular in a region with magnetic charges.

Since the solutions of the problems mentioned are to be expressed as series in $(x^1)^\alpha$, the metric tensor elements g_{ik} and $\sqrt{g}H^2$, $\sqrt{g}H^3$ will be written in the same form,

$$g_{11} = \sum_{k=0}^{\infty} a_k (x^1)^k, \quad g_{22} = \sum_{k=0}^{\infty} b_k (x^1)^k, \quad g_{33} = \sum_{k=0}^{\infty} c_k (x^1)^k,$$
$$\sqrt{g} H^2 = \sum_{k=0}^{\infty} H_k (x^1)^k, \quad \sqrt{g} H^3 = \sum_{k=0}^{\infty} h_k (x^1)^k. \qquad (1.4)$$

The indices k under the summation signs have the usual (not tensor) meaning of numbering the terms in the series and indicating powers. For convenience, the following notation is also introduced for the coefficients of the expansions of the elements g^{ik}, $\sqrt{g}$ and their combinations $\sqrt{g}\, g^{ik}$:

$$g^{11} = \sum_{k=0}^{\infty} A_k (x^1)^k, \quad g^{22} = \sum_{k=0}^{\infty} B_k (x^1)^k,$$
$$g^{33} = \sum_{k=0}^{\infty} C_k (x^1)^k, \quad \sqrt{g} = \sum_{k=0}^{\infty} G_k (x^1)^k,$$
$$\sqrt{g} g^{11} = \sum_{k=0}^{\infty} \alpha_k (x^1)^k, \quad \sqrt{g} g^{22} = \sum_{k=0}^{\infty} \beta_k (x^1)^k,$$
$$\sqrt{g} g^{33} = \sum_{k=0}^{\infty} \gamma_k (x^1)^k. \qquad (1.5)$$

§2. **Emission in the ρ-Mode with $H \neq 0$** is defined by the following conditions on the emitter: when $x^1 = 0$,

$$V = 0, \ \varphi = 0, \ \partial\varphi / \partial x^1 = 0, \qquad \rho v_{x^1} = J(x^2, x^3),$$
$$H_{x^1} = 0, \qquad H_{x^2} = m(x^2, x^3), \qquad H_{x^3} = n(x^2, x^3), \qquad (2.1)$$
$$m^2 + n^2 = h^2,$$

where v_{x^1}, H_{x^i} are the physical components of velocity and magnetic field. The solution of problem (1.1) and

(2.1) will be sought as

$$v_1 = \sum_{k=2}^{\infty} U_k (x^1)^{1/3 k}, \quad v_2 = x^1 \sum_{k=0}^{\infty} V_k (x^1)^{1/3 k},$$

$$v_3 = x^1 \sum_{k=0}^{\infty} W_k (x^1)^{1/3 k},$$

$$2\varphi = \sum_{k=4}^{\infty} \varphi_k (x^1)^{1/3 k}, \qquad 2\sqrt{g}\rho = \sum_{k=-2}^{\infty} \rho_k (x^1)^{1/3 k} \qquad (2.2)$$

The conditions that the flow be regular lead to

$$V_k = \frac{3}{k+3}(U_k)_2', \quad W_k = \frac{3}{k+3}(U_k)_3' \qquad (k \neq 3q,\ k \geqslant 2),$$

$$V_{3q} = \frac{1}{q+1}[(U_{3q})_2' - h_q], \quad W_{3q} = \frac{1}{q+1}[(U_{3q})_3' + H_q] \qquad (q = 0, 1, \ldots), \qquad (2.3)$$

$$V_1 = W_1 = 0, \qquad (U_k)_2' = \frac{\partial U_k}{\partial x^2}, \qquad (U_k)_3' = \frac{\partial U_k}{\partial x^3}.$$

Substitution for v_i in the first of Eqs. (1.1) gives

$$\varphi_s = \sum_{k=2}\Big[\Big(U_k^2 + 2\sum_{l=2}^{k-1} U_l U_{2k-l}\Big) A_{1/3(s-2k)} + \Big(2\sum_{l=2}^{k} U_l U_{2k-l+1}\Big) A_{1/3(s-2k-1)}\Big] +$$
$$+ \sum_{k=0}\Big[\Big(V_k^2 + 2\sum_{l=0}^{k-1} V_l V_{2k-l}\Big) B_{1/3(s-2k-6)} + \Big(2\sum_{l=0}^{k} V_l V_{2k-l+1}\Big) B_{1/3(s-2k-7)} +$$
$$+ \Big(W_k^2 + 2\sum_{l=0}^{k-1} W_l W_{2k-l}\Big) C_{1/3(s-2k-6)} + \Big(2\sum_{l=0}^{k} W_l W_{2k-l+1}\Big) C_{1/3(s-2k-7)}\Big], \qquad (2.4)$$
$$(s = 4, 5, \ldots).$$

The summation over k is controlled via the fractional subscript in A, B and C. For instance, with s = 7 the first group of terms in (2.4) gives a term with k = 2, the second with k = 3, the fourth and sixth with k = 0, and the third and fifth play no part at all; the coefficients with indices that are absent from (1.4), (1.5), and (2.2) are zero by definition (e. g., all the coefficients with negative indices in (1.4) and (1.5)). It is also assumed conventionally that summation over k from *a* to b with b < *a* gives zero.

Using the Poisson equation we get

$$\rho_{t-3} = \frac{t}{9}\sum_{s=0}^{t-1}(s+4)\varphi_{s+4}\alpha_{1/3(t-s-1)} + \sum_{s=4}^{t-3}\{[(\varphi_s)_2'\beta_{1/3(t-s-3)}]_2' + [(\varphi_s)_3'\gamma_{1/3(t-s-3)}]_3'\} \qquad (2.5)$$
$$(t = 1, 2, \ldots).$$

Finally, the equation for conservation of current provides relationships for determining the functions $U_k(x^2, x^3)$:

$$\frac{1}{3}p\sum_{t=0}^{p}\rho_{t-2}\sum_{l=0} A_l U_{p-t-3l-2} + \sum_{t=0}^{p-4}\Big[\Big(\rho_{t-2}\sum_{l=0} B_l V_{p-t-3l-4}\Big)_2' + \Big(\rho_{t-2}\sum_{l=0} C_l W_{p-t-3l-4}\Big)_3'\Big] = 0 \quad (p = 1, 2, \ldots). \qquad (2.6)$$

Formulas (2.2)–(2.6) embody the solution of the problem. Taking (2.6) with p = 1, we get $U_3 = \varphi_5 = \rho_{-1} = 0$. Writing (2.6) with p = 2, 3, 4, and 5 and recalling that the emission current density is given, the following terms of the potential expansion are obtained:

$$\varphi_4 = \Big(\frac{9}{2}J\Big)^{2/3} a_0^{2/3}, \quad \varphi_5 = 0, \quad \varphi_6 = \frac{1}{10}h^2 a_0,$$

$$\varphi_7 = \Big(\frac{9}{2}J\Big)^{2/3}\Big(\frac{1}{3}\frac{a_1}{a_0^{3/2}} + \frac{8}{15}T\Big)a_0^{7/6},$$

$$\varphi_8 = \frac{1}{14}\Big(\frac{9}{2}J\Big)^{1/3}\Big[\frac{9}{100}\frac{h^4}{J} + \frac{2}{7}\frac{nJ_P' - mJ_Q'}{J} - (nk_2 - m\delta_2) + (n_P' - m_Q')\Big]a_0^{4/3},$$

$$\varphi_9 = \Big[\frac{1}{20}\frac{a_1}{a_0^{3/2}}h^2 + \frac{27}{700}h^2 T + \frac{1}{28}(\varkappa_1 n^2 + \varkappa_2 m^2) + \frac{1}{56}(h^2)_S'\Big]a_0^{3/2},$$

where the subscripts S, P, Q denote differentiation with respect to arc along the curvilinear axes x^1, x^2, x^3; $\varkappa_1$ and $\varkappa_2$ are the principal curvatures of the surface $x^1 = 0$, $T = \varkappa_1 + \varkappa_2$ is its total curvature; k_1 and k_2, δ_1 and δ_2 are the principal curvatures of the surfaces $x^2 = \text{const}$, $x^3 = \text{const}$ respectively, evaluated at $x^1 = 0$.

If the arc length S along the x^1 curvilinear axis orthogonal to the emitter is taken as the expansion parameter, the expression for the potential becomes

$$2\varphi = \Big(\frac{9}{2}J\Big)^{2/3}S^{4/3} + \frac{1}{10}h^2S^2 + \frac{8}{15}\Big(\frac{9}{2}J\Big)^{2/3}TS^{7/3} + \frac{1}{14}\Big(\frac{9}{2}J\Big)^{1/3}\Big[\frac{9}{100}\frac{h^4}{J} + \frac{2}{7}\frac{nJ_P' - mJ_Q'}{J} - (nk_2 - m\delta_2) + (n_P' - m_Q')\Big]S^{8/3} +$$
$$+ \Big[\frac{27}{700}h^2T + \frac{1}{28}(\varkappa_1 n^2 + \varkappa_2 m^2) + \frac{1}{56}(h^2)_S'\Big]S^3 + \ldots.$$

The first correction to the Child-Langmuir three-halves power law in the local form depends only on the absolute value of the magnetic field strength at the emitter. The next term is the same as the first correction to the three-halves power law in the electrostatic case. The fourth term represents subtler effects, due not only to the magnetic field, but also to inhomogeneities in the field and in the emission current density, and to geometric factors. Finally, the coefficient of S^3 takes account of the magnetic field interaction with the emitter geometry and the rate of change of h^2 in the direction of the normal to $x^1 = 0$.

§3. Emission in the T-mode in a given external magnetic field H is determined by the following conditions: with $x^1 = 0$,

$$V = 0, \qquad \varphi = 0, \qquad \sqrt{g^{11}}\,\partial\varphi/\partial x^1 = \varepsilon(x^2, x^3),$$

$$\rho v_{x^1} = J(x^2, x^3), \qquad H_{x^1} = 0, \qquad H_{x^2} = m(x^2, x^3), \quad (3.1)$$
$$H_{x^3} = n(x^2, x^3), \qquad m^2 + n^2 = h^2:$$

Expansions for v_i, φ, ρ satisfying (1.1) and (3.1) will be obtained in half-integer powers of x^1

$$v_1 = \sum_{k=1}^{\infty} U_k (x^1)^{1/2 k}, \qquad v_2 = x^1 \sum_{k=0}^{\infty} V_k (x^1)^{1/2 k},$$
$$v_3 = x^1 \sum_{k=0}^{\infty} W_k (x^1)^{1/2 k},$$
$$2\varphi = \sum_{k=2}^{\infty} \varphi_k (x^1)^{1/2 k}, \qquad 2\sqrt{g}\,\rho = \sum_{k=-1}^{\infty} \rho_k (x^1)^{1/2 k}. \quad (3.2)$$

The conditions for the generalized momentum to be potential lead to the following relations between V_k, W_k and U_k, H_k, h_k:

$$V_{2q} = \frac{1}{q+1}[(U_{2q})_2' - h_q], \qquad V_{2q+1} = \frac{1}{q + 3/2}(U_{2q+1})_2',$$
$$V_0 = -h_0 \qquad W_{2q} = \frac{1}{q+1}[(U_{2q})_3' + H_q], \quad (3.3)$$
$$W_{2q+1} = \frac{1}{q + 3/2}(U_{2q+1})_3', \qquad W_0 = H_0 \qquad (q = 0, 1, \ldots).$$

The energy integral is used to find the coefficients of the potential expansion,

$$\varphi_s = \sum_{k=1}\Big[\Big(U_k^2 + 2\sum_{l=1}^{k-1} U_l U_{2k-l}\Big) A_{1/2 s-k} + \Big(2\sum_{l=1}^{k} U_l U_{2k-l+1}\Big) A_{1/2 (s-1)-k}\Big] +$$
$$+ \sum_{k=0}\Big[\Big(V_k^2 + 2\sum_{l=0}^{k-1} V_l V_{2k-l}\Big) B_{1/2 (s-4)-k} + \Big(2\sum_{l=0}^{k} V_l V_{2k-l+1}\Big) B_{1/2 (s-5)-k} +$$
$$+ \Big(W_k^2 + 2\sum_{l=0}^{k-1} W_l W_{2k-l}\Big) C_{1/2 (s-4)-k} + \Big(2\sum_{l=0}^{k} W_l W_{2k-l+1}\Big) C_{1/2 (s-5)-k}\Big] \qquad (s = 2, 3, \ldots). \quad (3.4)$$

Poisson's equation gives for ρ_{t-2}

$$\rho_{t-2} = \sum_{s=2} \{{}^1/_4 st \varphi_s \alpha_{1/2 (t-s)+1} + [(\varphi_s)_2' \beta_{1/2 (t-s)-1}]_2' + [(\varphi_s)_3' \gamma_{1/2 (t-s)-1}]_3'\} \quad (t = 1, 2, \ldots). \quad (3.5)$$

Finally, recurrence relations are obtained for U_k from the equation for conservation of current,

$$\sum_{l=1}\Big\{ {}^1/_2 p\, \rho_{l-2} \sum_{l=1} U_l A_{1/2 (p-l-l)+1} + \Big[\rho_{l-2}\sum_{l=0} V_l B_{1/2 (p-l-l)-1}\Big]_2' + \Big[\rho_{l-2}\sum_{l=0} W_l C_{1/2 (p-l-l)-1}\Big]_3'\Big\} = 0 \quad (p = 1, 2, \ldots). \quad (3.6)$$

Formulas (3.2)–(3.6) provide the solution of the problem.

The first coefficients in the potential expansion are

$$\varphi_2 = 2\varepsilon a_0^{1/2}, \qquad \varphi_3 = \frac{4\sqrt{2}}{3}\frac{J}{\sqrt{\varepsilon}} a_0^{3/4},$$
$$\varphi_4 = \Big(\frac{1}{2}\frac{a_1}{a_0^{3/2}}\varepsilon + \varepsilon T - \frac{1}{3}\frac{J^2}{\varepsilon^2}\Big) a_0,$$
$$\varphi_5 = \sqrt{2}\frac{J}{\sqrt{\varepsilon}}\Big(\frac{1}{2}\frac{a_1}{a_0^{3/2}} + \frac{11}{15}T + \frac{1}{9}\frac{J^3}{\varepsilon^3} + \frac{1}{15}\frac{Jh^2}{\varepsilon}\Big) a_0^{5/4},$$
$$\varphi_6 = \Big\{\frac{1}{2}\frac{a_1}{a_0^{3/2}}\Big(\varepsilon T - \frac{1}{3}\frac{J^2}{\varepsilon^2}\Big) - \frac{1}{12}\frac{a_1^2}{a_0^3}\varepsilon + \frac{1}{3}\frac{a_2}{a_0^2}\varepsilon - \frac{1}{5}\frac{J^2}{\varepsilon^2}T + \frac{1}{3}\varepsilon T^2 + \frac{1}{3}\varepsilon T_S' - \frac{8}{81}\frac{J^4}{\varepsilon^5} -$$
$$- \frac{4}{45}\frac{J^2h^2}{\varepsilon^3} + \frac{1}{9\varepsilon}\Big[J(m\delta_2 - nk_2) + (nJ)_P' - (mJ)_Q' + \frac{J}{\varepsilon}(-n\varepsilon_P' + m\varepsilon_Q')\Big] +$$
$$+ \frac{1}{3}[(3k_1 + k_2)\varepsilon_P' + (3\delta_1 + \delta_2)\varepsilon_Q' - 2\varepsilon(k_1^2 + \delta_1^2) - \varepsilon(k_1k_2 + \delta_1\delta_2) - \varepsilon_P'' - \varepsilon_Q'' + \varepsilon(k_{1P}' + \delta_{1Q}')].$$

The expansion in $s = a_0^{1/2} x^1$ can be transformed into an expansion in the arc length S along the x^1 axis by means of

$$s = S - \frac{1}{4}\frac{a_1}{a_0^{3/2}} S^2 + \frac{1}{6}\Big(\frac{a_1^2}{a_0^3} - \frac{a_2}{a_0^2}\Big) S^3 + \Big(-\frac{7}{48}\frac{a_1^3}{a_0^{9/2}} + \frac{13}{48}\frac{a_1 a_2}{a_0^{7/2}} - \frac{1}{8}\frac{a_3}{a_0^{5/2}}\Big) S^4 + \ldots \quad (3.7)$$

This gives

$$2\varphi = 2\varepsilon S + \frac{4\sqrt{2}}{3}\frac{J}{\sqrt{\varepsilon}} S^{3/2} + \Big(\varepsilon T - \frac{1}{3}\frac{J^2}{\varepsilon^2}\Big) S^2 + \frac{\sqrt{2}}{15}\frac{J}{\varepsilon\sqrt{\varepsilon}}\Big(11\varepsilon T + \frac{5}{3}\frac{J^2}{\varepsilon^2} + h^2\Big) S^{5/2} + \Big\{-\frac{1}{5}\frac{J^2}{\varepsilon^2} T + \frac{1}{3}\varepsilon T^2 + \frac{1}{3}\varepsilon T_S' -$$
$$- \frac{8}{81}\frac{J^4}{\varepsilon^5} - \frac{4}{45}\frac{J^2h^2}{\varepsilon^3} + \frac{1}{9\varepsilon}\Big[J(m\delta_2 - nk_2) + (nJ)_P' - (mJ)_Q' + \frac{J}{\varepsilon}(-n\varepsilon_P' + m\varepsilon_Q')\Big] +$$
$$+ \frac{1}{3}[(3k_1 + k_2)\varepsilon_P' + (3\delta_1 + \delta_2)\varepsilon_Q' - 2\varepsilon(k_1^2 + \delta_1^2) - \varepsilon(k_1k_2 + \delta_1\delta_2) - \varepsilon_P'' - \varepsilon_Q'' + \varepsilon(k_{1P}' + \delta_{1Q}')]\Big\} S^3 + \ldots \quad (3.8)$$

Comparing (2.7) with (3.8), it can be seen that when $\varepsilon \neq 0$ the action of the magnetic field is hampered, while geometric effects predominate: h^2 appears in the coefficient of $S^{5/2}$, and T in that of S^2; the magnetic field gradients only appear in the last term of (3.8), which also takes account of the variation in the total curvature along the normal to the emitter [in (2.7) the latter only affects the coefficient of $S^{10/3}$], etc.

§4. **The case of nonzero initial velocity** implies the following conditions on the emitting surface: with $x^1 = 0$,

$$v_{x^1} = u = \text{const}, \quad \varphi = 0, \quad \sqrt{g^{11}}\,\partial\varphi/\partial x^1 = \varepsilon(x^2, x^3),$$
$$\rho v_{x^1} = J(x^2, x^3), \qquad H_{x^1} = 0, \qquad H_{x^2} = m(x^2, x^3),$$
$$H_{x^3} = n(x^2, x^3), \qquad m^2 + n^2 = h^2. \quad (4.1)$$

The solution of problem (1.1) and (4.1) will be sought as

$$v_1 = \sum_{k=0}^{\infty} U_k (x^1)^k, \qquad v_2 = x^1 \sum_{k=0}^{\infty} V_k (x^1)^k,$$
$$v_3 = x^1 \sum_{k=0}^{\infty} W_k (x^1)^k,$$
$$2\varphi + (u)^2 = \sum_{k=0}^{\infty} \varphi_k (x^1)^k, \qquad 2\sqrt{g}\,\rho = \sum_{k=0}^{\infty} \rho_k (x^1)^k. \quad (4.2)$$

The dependences of V_k, W_k on U_k and the coefficients of the magnetic field component expansions are given by

$$(k+1) V_k = (U_k)_2' - h_k,$$
$$(k+1) W_k = (U_k)_3' + H_k \quad (k = 0, 1, \ldots), \quad (4.3)$$

where $u = A_0^{1/2} U_0$. Using the energy integral, we get

$$\begin{aligned}\varphi_s = \sum_{k=0}\Big[\Big(U_k^2 &+ 2\sum_{l=1}^{k} U_{l-1}U_{2k-l+1}\Big)A_{s-2k} + \Big(2\sum_{l=0}^{k} U_l U_{2k-l+1}\Big)A_{s-2k-1} + \\ &+\Big(V_k^2 + 2\sum_{l=1}^{k} V_{l-1}V_{2k-l+1}\Big)B_{s-2k-2} + \\ &+\Big(2\sum_{l=0}^{k} V_l V_{2k-l+1}\Big)B_{s-2k-3} + \\ &+\Big(W_k^2 + 2\sum_{l=1}^{k} W_{l-1}W_{2k-l+1}\Big)C_{s-2k-2} + \\ &+\Big(2\sum_{l=0}^{k} W_l W_{2k-l+1}\Big)C_{s-2k-3}\Big] \qquad (s = 0,\ 1, \ldots). \qquad (4.4)\end{aligned}$$

Poisson's equation gives for ρ_{t-1}

$$\rho_{t-1} = t\sum_{s=1}^{t+1} s\varphi_s \alpha_{t-s+1} + \sum_{s=1}^{t-1}\{[(\varphi_s)_2{}'\beta_{t-s-1}]_2{}' + [(\varphi_s)_3{}'\gamma_{t-s-1}]_3{}'\} \qquad (t = 1,\ 2, \ldots). \qquad (4.5)$$

The recurrence relations for the expansion coefficients are

$$p\sum_{t=0}^{p}\rho_t\sum_{l=0}^{p-t} A_l U_{p-t-l} + \sum_{t=0}^{p-2}\Big[\Big(\rho_t\sum_{l=0}^{p-t-2} B_l V_{p-t-l-2}\Big)_2{}' + \Big(\rho_t\sum_{l=0}^{p-t-2} C_l W_{p-t-l-2}\Big)_3{}'\Big] = 0 \qquad (p = 1,\ 2, \ldots). \qquad (4.6)$$

To assess the effect of the magnetic field, we write down the first four terms of the potential expansion

$$\varphi_1 = 2\varepsilon a_0^{1/2}, \qquad \varphi_2 = \Big(\frac{1}{2}\frac{a_1}{a_0^{3/2}}\varepsilon + \frac{J}{u} + \varepsilon T\Big)a_0,$$

$$\begin{aligned}\varphi_3 = \Big[&\frac{1}{2}\frac{a_1}{a_0^{3/2}}\Big(\frac{J}{u} + \varepsilon T\Big) - \frac{1}{12}\frac{a_1^2}{a_0^3}\varepsilon + \frac{1}{3}\frac{a_2}{a_0^2}\varepsilon + \\ &+\frac{1}{3}\Big(-\frac{\varepsilon J}{u^3} + 2\frac{J}{u}T + \varepsilon T^2 + \varepsilon T_S{}'\Big) - \frac{2}{3}\varepsilon(k_1^2 + \delta_1^2) - \\ &-\frac{1}{3}\varepsilon(k_1k_2 + \delta_1\delta_2) + \Big(k_1 + \frac{1}{3}k_2\Big)\varepsilon_P{}' + \\ &+\Big(\delta_1 + \frac{1}{3}\delta_2\Big)\varepsilon_Q{}' - \frac{1}{3}(\varepsilon_P{}'' + \varepsilon_Q{}'') + \frac{1}{3}\varepsilon(k_{1P}{}' + \delta_{1Q}{}')\Big]a_0^{3/2},\end{aligned}$$

$$\begin{aligned}\varphi_4 = \Big\{&\frac{a_1}{a_0^{3/2}}\Big[-\frac{1}{4}\frac{\varepsilon J}{u^3} + \frac{1}{2}\frac{J}{u}T + \frac{1}{4}\varepsilon T^2 + \frac{1}{4}\varepsilon T_S{}' + \\ &+\frac{3}{4}\Big(-\frac{2}{3}\varepsilon(k_1^2 + \delta_1^2) - \frac{1}{3}\varepsilon(k_1k_2 + \delta_1\delta_2) + \\ &+\Big(k_1 + \frac{1}{3}k_2\Big)\varepsilon_P{}' + \Big(\delta_1 + \frac{1}{3}\delta_2\Big)\varepsilon_Q{}' - \frac{1}{3}(\varepsilon_P{}'' + \varepsilon_Q{}'') + \\ &+\frac{1}{3}\varepsilon(k_{1P}{}' + \delta_{1Q}{}')\Big)\Big] - \frac{1}{48}\frac{a_1^2}{a_0^3}\Big(\frac{J}{u} + \varepsilon T\Big) + \frac{1}{32}\frac{a_1^3}{a_0^{9/2}}\varepsilon + \\ &+\frac{1}{3}\frac{a_2}{a_0^2}\Big(\frac{J}{u} + \varepsilon T\Big) - \\ &-\frac{1}{8}\frac{a_1a_2}{a_0^{7/2}}\varepsilon + \frac{1}{4}\frac{a_3}{a_0^{5/2}}\varepsilon - \frac{1}{12}\frac{J^2}{u^4} + \frac{1}{4}\frac{\varepsilon^2 J}{u^5} + \frac{1}{12}\frac{Jh^2}{u^3} - \frac{1}{3}\frac{\varepsilon J}{u^3}T + \\ &+\frac{1}{4}\frac{J}{u}T^2 + \frac{1}{12}\varepsilon T^3 + \frac{1}{4}\frac{J}{u}T_S{}' + \frac{1}{4}\varepsilon T T_S{}' + \frac{1}{12}\varepsilon T_S{}'' + \\ &+\frac{1}{12}\frac{J}{u^2}(m\delta_2 - nk_2) + \frac{3}{4}T\Big[-\frac{2}{3}\varepsilon(k_1^2 + \delta_1^2) - \\ &-\frac{1}{3}\varepsilon(k_1k_2 + \delta_1\delta_2) + \Big(k_1 + \frac{1}{3}k_2\Big)\varepsilon_P{}' + \Big(\delta_1 + \frac{1}{3}\delta_2\Big)\varepsilon_Q{}' - \\ &-\frac{1}{3}(\varepsilon_P{}'' + \varepsilon_Q{}'') + \frac{1}{3}\varepsilon(k_{1P}{}' + \delta_{1Q}{}')\Big] - \\ &-\Big[\frac{1}{3}(\varkappa_{1P}{}' - k_{1S}{}') - \frac{3}{4}\Big(\varkappa_1 - \frac{1}{3}\varkappa_2\Big)\Big(k_1 + \frac{1}{3}k_2\Big)\Big]\varepsilon_P{}' - \\ &-\Big[\frac{1}{3}(\varkappa_{2Q}{}' - \delta_{1S}{}') + \frac{3}{4}\Big(\frac{1}{3}\varkappa_1 - \varkappa_2\Big)\Big(\delta_1 + \frac{1}{3}\delta_2\Big)\Big]\varepsilon_Q{}' - \\ &-\frac{1}{2}\varepsilon k_1\Big[k_{1S}{}' - \Big(\varkappa_1 + \frac{1}{2}\varkappa_2\Big)_P{}' + \varkappa_1\Big(k_1 + \frac{1}{2}k_2\Big)\Big] - \\ &-\frac{1}{2}\varepsilon\delta_1\Big[\delta_{1S}{}' - \Big(\frac{1}{2}\varkappa_1 + \varkappa_2\Big)_Q{}' + \varkappa_2\Big(\delta_1 + \frac{1}{2}\delta_2\Big)\Big] - \\ &-\frac{1}{4}\Big(\varkappa_1 - \frac{1}{3}\varkappa_2\Big)\varepsilon_P{}'' + \frac{1}{4}\Big(\frac{1}{3}\varkappa_1 - \varkappa_2\Big)\varepsilon_Q{}'' + \\ &+\frac{1}{4}\varepsilon\Big[\frac{1}{3}(k_{1S}{}' - T_P{}')_P{}' + \varkappa_1 k_{1P}{}' - \frac{1}{3}k_2(k_{1S}{}' - T_P{}')\Big] + \\ &+\frac{1}{4}\varepsilon\Big[\frac{1}{3}(\delta_{1S}{}' - T_Q{}')_Q{}' + \varkappa_2\delta_{1Q}{}' - \frac{1}{3}\delta_2(\delta_{1S}{}' - T_Q{}')\Big] - \\ &-\frac{1}{12u}\Big(J_P{}' - \frac{nJ}{u}\Big)_P{}' - \frac{1}{12u}\Big(J_Q{}' + \frac{mJ}{u}\Big)_Q{}' + \\ &+\frac{1}{2u}J_P{}'\Big(k_1 + \frac{1}{6}k_2\Big) + \frac{1}{2u}J_Q{}'\Big(\delta_1 + \frac{1}{6}\delta_2\Big) - \\ &-\frac{J}{u}\Big[\frac{7}{12}(k_1^2 + \delta_1^2) + \frac{1}{4}(k_1k_2 + \delta_1\delta_2) - \\ &-\frac{1}{4}(k_{1P}{}' + \delta_{1Q}{}')\Big]\Big\}a_0^2.\end{aligned}$$

Using (3.7), we arrive at the expression

$$\begin{aligned}2\varphi = -u^2 + 2\varepsilon S &+ \Big(\frac{J}{u} + \varepsilon T\Big)S^2 + \\ &+\frac{1}{3}\Big[-\frac{\varepsilon J}{u^3} + 2\frac{J}{u}T + \varepsilon T^2 + \varepsilon T_S{}' - \\ &-2\varepsilon(k_1^2 + \delta_1^2) - \varepsilon(k_1k_2 + \delta_1\delta_2) + \\ &+(3k_1 + k_2)\varepsilon_P{}' + (3\delta_1 + \delta_2)\varepsilon_Q{}' - \varepsilon_P{}'' - \varepsilon_Q{}'' + \\ &+\varepsilon(k_{1P}{}' + \delta_{1Q}{}')\Big]S^3 + \vartheta_4 S^4 + \ldots. \qquad (4.7)\end{aligned}$$

Here, ϑ_4 is the part of φ_4 not containing a_1, a_2, a_3. A nonzero initial velocity of emission leads to the same sort of trend as a nonzero electric field, i.e., geometric factors predominate over the magnetic field. For, with $\varepsilon = 0$, up to terms corresponding to uniform flow between parallel planes ($T = 0$), the third term in the expansion is $2/3\ u^{-1}JT$, and the magnetic field only appears first in the coefficient of S^4. The contribution of H to φ_4 is the result, not only of the absolute value of the magnetic field at the emitter, but also of its derivatives in the directions P, Q. When $\varepsilon \neq 0$ the geometric effects will be even more marked, since the total curvature now first appears in φ_2.

The problems discussed above, together with those dealt with in [1], cover the whole range of problems that can be devised for a single-energy beam when all the necessary conditions are specified on the emitting surface.

§5. Beam in the presence of a moving background. The results of §§2–4 are easily extended to the case of flow against a moving background of uniform density N_0, where $N_0 > 0$ if the background is made up of particles with the same charge sign as the beam particles, and $N_0 < 0$ in the opposite case. The solution is given by the formulas of §§2–4, except that the functions ρ_t in (2.6), (3.6) and (4.6) have to be replaced by functions R_t given by the formulas below.

Denoting by Φ_k the coefficients of the potential expansion in the presence of the background, and by Δ_k the corrections to the functions φ_k obtained above, corresponding to the case $N_0 = 0$, $\Delta_k = \Phi_k - \varphi_k$, the formulas for the corrections are as follows.

With emission in the ρ-mode,

$$R_{3q} = \rho_{3q} - 2N_0G_q \quad (q = 0,\ 1, \ldots),$$

$$R_t = \rho_t \quad (t \neq 3q), \qquad \Delta_4 = \Delta_5 = \Delta_7 = 0,$$

$$\Delta_6 = \frac{9}{10}N_0 a_0, \qquad \Delta_8 = \frac{9}{200}\Big(\frac{4}{3}\Big)^{1/3}J^{-2/3}N_0\Big(h^2 - \frac{17}{14}N_0\Big)a_0^{4/3},$$

$$\Delta_9 = N_0\Big(\frac{9}{20}\frac{a_1}{a_0^{3/2}} + \frac{99}{350}T\Big)a_0^{3/2}.$$

Denoting by ϑ_k the coefficients of the potential expansion in the variable S in the case $N_0 = 0$, we have

$$2\varphi = \varphi_4 S^{4/3} + \left(\vartheta_6 + \frac{9}{10} N_0\right) S^2 + \vartheta_7 S^{7/3} +$$
$$+ \left[\vartheta_8 + \frac{9}{200}\left(\frac{4}{3}\right)^{1/3} J^{-2/3} N_0 \left(h^2 - \frac{17}{14} N_0\right)\right] S^{8/3} +$$
$$+ \left(\vartheta_9 + \frac{99}{350} N_0 T\right) S^3 + \ldots.$$

It is clear from this that the background produces the same type of effect as a nonzero magnetic field: when $H = 0$, but $N_0 \neq 0$, the expansion is in powers of $S^{1/3}$.

For emission in the T-mode,

$$R_{2q} = \rho_{2q} - 2N_0 G_q, \qquad R_{2q-1} = \rho_{2q-1} \qquad (q = 0, 1, \ldots);$$

$$\Delta_2 = \Delta_3 = 0, \quad \Delta_4 = N_0 a_0, \quad \Delta_5 = -\frac{2\sqrt{2}}{45} \frac{J}{\varepsilon\sqrt{\varepsilon}} N_0 a_0^{5/4},$$

$$\Delta_6 = \left(\frac{1}{2} \frac{a_1}{a_0^{5/3}} + \frac{70}{9} \frac{J^2}{\varepsilon^3} + \frac{1}{3} T\right) N_0 a_0^{3/2},$$

$$2\varphi = \varphi_2 S + \varphi_3 S^{5/2} + (\vartheta_4 + N_0) S^2 + \left(\vartheta_5 - \frac{2\sqrt{2}}{45} \frac{J}{\varepsilon\sqrt{\varepsilon}} N_0\right) S^{5/2} +$$
$$+ \left[\vartheta_6 + \left(\frac{70}{9} \frac{J^2}{\varepsilon^3} + \frac{1}{3} T\right) N_0\right] S^3 + \ldots.$$

For nonzero initial velocity,

$$R_t = \rho_t - 2N_0 G_t \qquad (t = 0, 1, \ldots),$$

$$\Delta_1 = \Delta_2 = 0, \quad \Delta_3 = \frac{1}{3}\left(\frac{\varepsilon}{u^2} - T\right) N_0 a_0^{3/2},$$

$$\Delta_4 = \left\{\frac{1}{4} \frac{a_1}{a_0^{3/2}}\left(\frac{\varepsilon}{u^2} - T\right) + \frac{1}{12} \frac{J}{u^3} - \frac{1}{4} \frac{\varepsilon^2}{u^4} - \frac{1}{12} \frac{h^2}{u^2} -\right.$$
$$- \frac{1}{3} \frac{\varepsilon}{u^2} T + \frac{1}{6} T^2 - \frac{1}{12} T_S' + \frac{1}{12}\left[-\left(k_1 + \frac{n}{u}\right)_P' - \left(\delta_1 - \frac{m}{u}\right)_Q' +\right.$$
$$\left.\left. + (2k_1 + k_2)\left(k_1 + \frac{n}{u}\right) + (2\delta_1 + \delta_2)\left(\delta_1 - \frac{m}{u}\right)\right]\right\} N_0 a_0^2,$$

$$2\varphi = \varphi_1 S + \varphi_2 S^2 + \left[\vartheta_3 + \frac{1}{3}\left(\frac{\varepsilon}{u^2} - T\right) N_0\right] S^3 +$$
$$+ \left\{\vartheta_4 + \left(\frac{1}{12} \frac{J}{u^3} - \frac{1}{4} \frac{\varepsilon^2}{u^4} - \frac{1}{12} \frac{h^2}{u^2} - \frac{1}{3} \frac{\varepsilon}{u^2} T + \frac{1}{6} T^2 - \frac{1}{12} T_S' +\right.\right.$$
$$+ \frac{1}{12}\left[-\left(k_1 + \frac{n}{u}\right)_P' - \left(\delta_1 - \frac{m}{u}\right)_Q' +\right.$$
$$\left.\left.\left. + (2k_1 + k_2)\left(k_1 + \frac{n}{u}\right) + (2\delta_1 + \delta_2)\left(\delta_1 - \frac{m}{u}\right)\right]\right)\right\} N_0 S^4 + \ldots.$$

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